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Diagonal lifts of tensor fields to the frame bundle of second order

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Introduction

Let M be an n-dimensional manifold of class C^∞, $\mathscr{F}M$ its frame bundle and $\mathscr{F}^2M$ its frame bundle of second order.

The purpose of the present paper is to introduce the (so called) diagonal lifts to $\mathscr{F}^2M$ of tensor fields on M of type $(0, s)$ or $(1, s)$, $s \geqq 1$, with respect to a connection of order 2 on M. A similar theory for $\mathscr{F}M$ and a linear connection on M has been developed by Cordero and one of us in [1]. Actually, the two theories are related.

The paper is structured as follows. In Sections 1 and 2 we recall, for later use, the definitions and properties of the frame bundle of second order $\mathscr{F}^2M$ and of connections of order 2 on M. In Section 3 we introduce a wide class of vector fields on $\mathscr{F}^2M$ and obtain some identities which will be very useful through the rest of the paper. Section 4 is devoted to the definition of the diagonal lift to $\mathscr{F}^2M$ of tensor fields of type $(0, s)$ or $(1, s)$, $s \geqq 1$, with respect to a connection of order 2 on M. The particular cases of the diagonal lifts of tensor fields of type $(1, 1)$ and $(0, 2)$ are the subject of Sections 5 and 6, respectively. We remark that polynomial structures lift into polynomial structures with the same structural polynomial and Riemannian metrics (resp., almost symplectic forms) lift into Riemannian metrics (resp., almost symplectic forms).

Finally, we apply the previous results of Sections 5 and 6 to show that the frame bundle of second order of an almost Hermitian manifold (M, J, G) admits an almost Hermitian structure (J^D, G^D); moreover, the Kaehler form of (J^D, G^D) is the diagonal lift to $\mathscr{F}^2M$ of the Kaehler form of (J, G) and the following result is easily proved: $(\mathscr{F}^2M, J^D, G^D)$ is never an almost Kaehler manifold.

Through the paper, manifolds, tensor fields and connections will be assumed differentiable of class C^∞.

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1. The frame bundle of order 2

In this section, we recall, for later use the definition and some properties of the frame bundle of order 2. More details can be found in [2], [3], [4], [6] and [7].

Let M be an n-dimensional manifold. If U and V are two neighborhoods of $0 \in R^n$, two mappings $f\colon U \to M$ and $g\colon V \to M$ are said to define the same r-jet at 0 if they have the same partial derivatives up to order r at 0. If f is a diffeomorphism of a neighborhood of 0 onto an open subset of M, then the r-jet $j^r f$ at 0 is called an r-frame at $x=f(0)$. Clearly, an 1-frame is an ordinary linear frame. The set of the r-frames of M, denoted by $\mathscr{F}^r M$, is a principal bundle over M with projection π^r, $\pi^r(j^r f)=f(0)$, and with structure group $G^r(n)$ which will be described next.

Let $G^r(n)$ be the set of r-frames $j^r g$ at $0 \in R^n$, where g is a diffeomorphism from a neighborhood of 0 in R^n onto a neighborhood of 0 in R^n. Then $G^r(n)$ is a Lie group with multiplication defined by the composition of jets, i.e.,

$$(j^r g)(j^r g') = j^r(g \circ g').$$

The group $G^r(n)$ acts of $\mathscr{F}^r M$ on the right by

$$(j^r f)(j^r g) = j^r(f \circ g) \quad \text{for} \quad j^r f \in \mathscr{F}^r M, \quad j^r g \in G^r(n).$$

Clearly, $\mathscr{F}^1 M$ is the bundle of linear frames over M with group $G^1(n)=\mathrm{Gl}(n)$ and projection $\pi^1=\pi$.

From now on, we shall only consider $\mathscr{F}^1 M$ and $\mathscr{F}^2 M$ and denote $\pi_1^2\colon \mathscr{F}^2 M \to \mathscr{F} M$, $\pi_1^2(j^2 f)=j^1 f$, the canonical projection.

For any coordinate system in M, (U, x^i), we consider the induced coordinate systems $\{\mathscr{F} U, (x^i, X^i_j)\}$ and $\{\mathscr{F}^2 U, (x^i, X^i_j, X^i_{jk})\}$ in $\mathscr{F} M$ and $\mathscr{F}^2 M$, respectively, where $X^i_{jk}=X^i_{kj}$.

We have a natural isomorphism $G^2(n) \cong \mathrm{Gl}(n) \times S^2(n)$, where $S^2(n)$ is the set of symmetric bilinear forms on R^n, multiplication on the right hand given by $(A, \alpha)(B, \beta)=(AB, \alpha \circ (B, B)+A \circ \beta)$.

Then, the Lie algebra $g^2(n)$ of $G^2(n)$ can be identified to $\mathrm{gl}(n) \oplus S^2(n)$, with a bracket product given by

(1.1)

$$[(A,\alpha), (B,\beta)] = \big([A, B], A \circ \beta - \beta \circ (I, A) - \beta \circ (A, I) - (B \circ \alpha - \alpha \circ (I, B) - \alpha \circ (B, I))\big)$$

where I is the unit matrix.

With these identifications, the adjoint representation of $G^2(n)$ in $a(n)=R^n \oplus \mathrm{gl}(n)$ is given by

$$\mathrm{Ad}^{(2)}(A, \alpha)(v, B) = \big(Av, \bar{\alpha}(v)A^{-1}+ABA^{-1}\big)$$

where $\bar{\alpha}\colon R^n \to \mathrm{gl}(n)$ is the linear map defined by $\bar{\alpha}(v)(w)=\alpha(v, w)$, and the adjoint representation of $G^2(n)$ in $g^2(n)=\mathrm{gl}\,(n)\oplus S^2(n)$ is given by

$$\mathrm{Ad}\,(A,\alpha)(B,\beta)=\big(ABA^{-1},\alpha\circ(A^{-1},BA^{-1})+\alpha\circ(BA^{-1},A^{-1})-$$
$$-ABA^{-1}\circ\alpha\circ(A^{-1},A^{-1})+A\circ\beta\circ(A^{-1},A^{-1})\big).$$

From now on, we shall denote by $\{E_i\}$, $\{E_j^i\}$ and $\{E_{jk}^i\}$, $i,j,k=1,\dots,n$; $E_{jk}^i=E_{kj}^i$; the canonical basis of R^n, $\mathrm{gl}\,(n)$ and $S^2(n)$, respectively.

Since $G^2(n)$ acts on $\mathscr{F}^2M$ on the right, every element (A,α) of the Lie algebra $g^2(n)$ of $G^2(n)$ induces a vector field $\lambda(A,\alpha)$ on $\mathscr{F}^2M$ called the *fundamental vector field corresponding to* (A,α). So, the vertical subspace at any point $p\in\mathscr{F}^2M$ can be decomposed as $\lambda(\mathrm{gl}\,(n))_p\oplus\lambda(S^2(n))_p$.

Let θ be the canonical form on $\mathscr{F}^2M$; θ is an $a(n)$-valued 1-form of type $\mathrm{Ad}^{(2)}(G^2(n))$ and satisfying $\theta(\lambda(A,\alpha))=A$. Let $\theta=\theta_{-1}+\theta_0$ be the decomposition of θ; then, θ_{-1} is an R^n-valued 1-form and θ_0 a $\mathrm{gl}\,(n)$-valued 1-form on $\mathscr{F}^2M$. We have

$$\theta_{-1}(\lambda(A,\alpha))=0,\quad \theta_0(\lambda(A,\alpha))=A.$$

Moreover, $\theta_{-1}=(\pi_1^2)^*\bar\theta$, where $\bar\theta$ is the canonical form of $\mathscr{F}M$. With respect to the canonical bases, we shall put

$$\theta_{-1}=\theta^iE_i,\quad \theta_0=\theta_j^iE_j^i,$$

where θ^i, θ_j^i are locally expressed in $\mathscr{F}^2M$ as

$$\theta^i=Y_k^i\,dx^k \tag{1.2}$$

$$\theta_j^i=Y_k^i(dX_j^k-X_{hj}^kY_l^h\,dx^l), \tag{1.3}$$

(Y_j^i) being the inverse matrix of (X_j^i). From (1.2) and (1.3), we easily obtain the following structure equation

$$d\theta^i=-\theta_k^i\Lambda\theta^k.$$

2. Connections of order 2

A connection Γ in the bundle $\mathscr{F}^2M$ of 2-frames of M is called a connection of order 2 on M.

Let ω be the connection form of Γ; then ω is an 1-form on $\mathscr{F}^2M$ of type $(\mathrm{Ad}\,(G^2(n)))$ and can be decomposed as follows:

$$\omega=\omega_0+\omega_1, \tag{2.1}$$

where ω_0 is a $\mathrm{gl}\,(n)$-valued, and ω_1 an $S^2(n)$-valued 1-form on $\mathscr{F}^2M$. Since $\omega(\lambda(A,\alpha))=(A,\alpha)$, we have

$$\omega_0(\lambda(A,\alpha))=A,\quad \omega_1(\lambda(A,\alpha))=\alpha.$$

Similarly, the curvature form Ω of Γ is a tensorial 2-form on $\mathscr{F}^2M$ of type Ad $(G^2(n))$ and can be decomposed as

$$\Omega = \Omega_0 + \Omega_1, \tag{2.2}$$

where Ω_0 (resp., Ω_1) is a gl(n)-valued (resp., $S^2(n)$-valued) 2-form on $\mathscr{F}^2M$. The structure equation is

$$d\omega = -(1/2)[\omega, \omega] + \Omega,$$

or

$$\begin{aligned} d\omega_0 &= -(1/2)[\omega_0, \omega_0] + \Omega_0, \\ d\omega_1 &= -(1/2)\{[\omega_0, \omega_1] + [\omega_1, \omega_0]\} + \Omega_1, \end{aligned} \tag{2.3}$$

taking into account (1.1), (2.1) and (2.2). With respect to the canonical bases, we can write

$$\omega_0 = \omega^i_j E^j_i, \quad \omega_1 = \omega^i_{jk} E^{jk}_i, \quad \Omega_0 = \Omega^i_j E^j_i, \quad \Omega_1 = \Omega^i_{jk} E^{jk}_i,$$

where $\omega^i_{jk} = \omega^i_{kj}$, $\Omega^i_{jk} = \Omega^i_{kj}$, and (2.3) can be equivalently written as

$$d\omega^i_j = -\omega^i_k \Lambda \omega^k_j + \Omega^i_j, \quad d\omega^i_{jk} = -\omega^i_r \Lambda \omega^r_{jk} + \omega^r_k \Lambda \omega^i_{jr} + \omega^r_j \Lambda \omega^i_{rk} + \Omega^i_{jk}.$$

We shall give the following definition.

Definition 2.1. A connection Γ of order 2 on M is said to be partially flat if $\Omega_1 = 0$.

Consequently, a flat connection of order 2 on M is always partially flat.

Let σ be the cross-section of $\mathscr{F}^2M$ over a coordinate neighborhood (U, x^i) which assigns to each $x \in U$ the 2-frame $(x^i, I, 0)$. We define functions Γ^i_{jk}, Γ^i_{jkl} on U, $\Gamma^i_{jkl} = \Gamma^i_{jlk}$, by

$$\sigma^* \omega_0 = (\Gamma^i_{jk}\, dx^j) E^k_i, \quad \sigma^* \omega_1 = (\Gamma^i_{jkl}\, dx^j) E^{kl}_i.$$

These functions Γ^i_{jk}, Γ^i_{jkl} are called *the components of the connection Γ with respect to the local coordinate system* (U, x^i). By a straightforward computation, we obtain

$$\omega^i_j = Y^i_k (dX^k_j + \Gamma^k_{ml} X^l_j\, dx^m),$$

$$\begin{aligned} \omega^i_{jk} = \{&-\Gamma^t_{ms}(X^s_k Y^c_t Y^i_p X^p_{jc} + X^s_j Y^c_t Y^i_p X^p_{ck} - X^s_{jk} Y^i_t) + \\ &+ \Gamma^t_{msl} X^s_j X^l_k Y^i_t\}\, dx^m - Y^t_r Y^i_s X^s_{tk}\, dX^r_j - Y^t_r Y^i_s X^s_{jt}\, dX^r_k + Y^i_s\, dX^s_{jk}. \end{aligned}$$

Moreover, if we put

$$\sigma^* \Omega_0 = (\sigma^* \Omega^i_j) E^j_i = (1/2)(R^i_{jkl}\, dx^k \Lambda\, dx^l) E^j_i,$$

$$\sigma^* \Omega_1 = (\sigma^* \Omega^i_{jk}) E^{jk}_i = (1/2)(R^i_{jktl}\, dx^t \Lambda\, dx^l) E^{jk}_i,$$

then, from (2.4), we obtain

$$R^i_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm},$$

$$R^i_{jktl} = \partial_t \Gamma^i_{ljk} - \partial_l \Gamma^i_{tjk} + \Gamma^r_{lk}\Gamma^i_{tjr} - \Gamma^r_{tk}\Gamma^i_{ljr} + \Gamma^r_{lj}\Gamma^i_{trk} - \Gamma^r_{tj}\Gamma^i_{lrk} + \Gamma^i_{tr}\Gamma^r_{ljk} - \Gamma^i_{lr}\Gamma^r_{tjk}.$$

We can easily prove that the $n+n^2+n^2(n+1)/2$ global 1-forms θ^i, ω^i_j, ω^i_{jk}, $j\leq k$, on $\mathscr{F}^2M$ are linearly independent everywhere.

Now, let $p\in\mathscr{F}^2M$; then $(\theta-1)_p$ gives a linear isomorphism of the horizontal subspace H_p at p onto R^n. Thus, we can associate with each $\xi\in R^n$ a horizontal vector field $C(\xi)$ on $\mathscr{F}^2M$ as follows. For each $p\in\mathscr{F}^2M$, $C(\xi)_p$ is the unique horizontal vector at p such that

$$(\theta-1)_p C(\xi)_p = \xi.$$

We call $C(\xi)$ *the standard horizontal vector field on $\mathscr{F}^2M$ corresponding to* ξ.

As a simple computation shows, the local expression of $C(\xi)$ in $\mathscr{F}^2U$ is

$$C(\xi) = X^i_m \xi^m \left\{ \frac{\partial}{\partial x^i} - \Gamma^k_{il} X^l_j \frac{\partial}{\partial X^k_j} - (\Gamma^s_{it} X^t_{jk} + \Gamma^s_{itl} X^t_j X^l_k) \frac{\partial}{\partial X^s_{jk}} \right\}, \tag{2.5}$$

if $\xi = \xi^i E_i$.

Remark that the n global vector fields $C(E_i)$ span the horizontal distribution H_Γ on $\mathscr{F}^2M$. So, the $n+n^2+n^2(n+1)/2$ global vector fields $C(E_i)$, λE^i_j, λE^i_{jk}, $j\leq k$, define a parallelism on $\mathscr{F}^2M$ and are dual to σ^i, ω^i_j, ω^i_{jk}; moreover the local expressions of λE^i_j, λE^i_{jk} on $\mathscr{F}^2U$ are

$$\lambda E^i_j = X^t_i \frac{\partial}{\partial X^t_j} + X^t_{is} \frac{\partial}{\partial X^t_{js}} + X^t_{si} \frac{\partial}{\partial X^t_{sj}}, \tag{2.6}$$

$$\lambda E^i_{jk} = X^t_i \frac{\partial}{\partial X^t_{jk}}. \tag{2.7}$$

From (2.5), (2.6) and (2.7), we notice that the horizontal distribution H_Γ is spanned by the local vector fields

$$D_i = \frac{\partial}{\partial x^i} - \Gamma^k_{il} X^l_r \frac{\partial}{\partial X^k_r} - \{\Gamma^s_{it} X^t_{rk} + \Gamma^s_{itl} X^t_r X^l_k\} \frac{\partial}{\partial X^s_{rk}}$$

and the vertical distribution V is spanned by the local vector fields

$$D^i_j = \frac{\partial}{\partial X^i_j} + Y^r_j X^t_{rs} \frac{\partial}{\partial X^t_{is}} + Y^r_j X^t_{sr} \frac{\partial}{\partial X^t_{si}}, \quad D^i_{jk} = \frac{\partial}{\partial X^i_{jk}}$$

when we restrict ourselves to $\mathscr{F}^2U$.

The frame $\{D_i, D^i_j, D^i_{jk}\}$ is adapted to the almost product structure $(H_\Gamma; V)$ and we call it the *adapted frame on $\mathscr{F}^2U$*. The local 1-forms η^i, η^i_j, η^i_{jk} on $\mathscr{F}^2U$ dual to $\{D^i, D^i_j, D^i_{jk}\}$ are given by

$$\eta^i = dx^i, \quad \eta^i_j = \Gamma^i_{rs} X^s_j \, dx^r + dX^i_j,$$

$$\eta^i_{jk} = \{(\Gamma^i_{tr} X^r_{jk} + \Gamma^i_{trs} X^r_j X^s_k) - Y^s_m \Gamma^m_{tr} (X^r_j X^i_{sk} + X^r_k X^i_{js})\} \, dx^t -$$
$$- Y^r_t (\delta^{sj} X^i_{rk} + \delta^{sk} X^i_{jr}) \, dX^t_s + dX^i_{jk},$$

and $\{\eta^i, \eta^i_j, \eta^i_{jk}\}$ will be called the *adapted coframe on* $\mathscr{F}^2 U$.

Now, let Γ be a connection of order 2 on M. Since the canonical projection π^2_1: $\mathscr{F}^2 M \to \mathscr{F} M$ is a homomorphism of principal bundles over the identity of M inducing the canonical projection $G^2(n) \to \mathrm{Gl}(n)$, then the connection Γ defines a connection in $\mathscr{F}M$, that is, a linear connection $\bar{\Gamma}$ on M. We call $\bar{\Gamma}$ the *linear connection on M induced from* Γ. If $\bar{\omega}$, $\bar{\Omega}$ are the connection and the curvature forms of $\bar{\Gamma}$, then

$$(2.8) \qquad (\pi^2_1)^* \bar{\omega} = \omega_0, \quad (\pi^2_1)^* \bar{\Omega} = \Omega_0.$$

Let $\bar{\lambda} A$ (resp., $B(\xi)$) be the fundamental vector field (resp., the standard horizontal vector field with respect to $\bar{\Gamma}$) corresponding to $A \in \mathrm{gl}(n)$ (resp., $\xi \in R^n$). A simple computation shows that

$$(2.9) \qquad \pi^2_1 \lambda(A, \alpha) = \bar{\lambda} A \quad (\text{resp., } \pi^2_1 C(\xi) = B(\xi)).$$

If $\bar{\sigma}$ is the cross section of $\mathscr{F}M$ over a coordinate neighborhood (U, x^i) which assigns to each $x \in U$ the linear frame (x^i, I), then we can define the components of $\bar{\Gamma}$ by

$$\sigma^* \bar{\omega} = (\bar{\Gamma}^i_{jk} \, dx^j) E^k_i.$$

Taking into account (2.8), we find $\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}$ and, consequently, $\bar{R}^i_{jkl} = R^i_{jkl}$, $\bar{R}^i_{jkl}$ being the components of the tensor curvature of $\bar{\Gamma}$.

3. Vector fields on $\mathscr{F}^2 M$

Our aim in this section is to introduce a wide class of vector fields on $\mathscr{F}^2 M$ and obtain some identities which will be very useful through the rest of the paper. Previously, we shall consider an arbitrary principal bundle $P(M, G)$ over M with structure group G. Let $\mathfrak{g}$ be the Lie algebra of G; then, for any function f: $P \to \mathfrak{g}$, we can define the vertical vector field λf by

$$(3.1) \qquad (\lambda f)(p) = (\lambda(f(p)))_p, \quad p \in P.$$

It will be called the *fundamental vector field corresponding to* f. On the other hand, if Γ is a connection in P, we shall denote by X^H the horizontal lift to P of a vector field X on M. Then, we have

Proposition 3.1. *Let Γ be a connection in P with connection form ω and curvature form Ω. Then*

(1) $$[\lambda f, \lambda g] = \lambda[f, g]+\lambda(\lambda f(g))-\lambda(\lambda g(f)),$$

(2) $$[X^H, \lambda g] = \lambda(X^H g),$$

(3) $$[X^H, Y^H] = [X, Y]^H-2\lambda\Omega(X^H, Y^H),$$

for any vector fields X, Y on M and any functions $f, g\colon P\to\mathfrak{g}$.

Proof. Let us recall the structure equation of Γ:

$$d\omega = -(1/2)[\omega, \omega]+\Omega.$$

Now, from (3.1), we have $\omega(\lambda f)=f$, $\omega(\lambda g)=g$ and, therefore,

$$2d\omega(\lambda f, \lambda g) = \lambda f(\omega(\lambda g))-\lambda g(\omega(\lambda f))-\omega([\lambda f, \lambda g]) = \\ = \lambda f(g)-\lambda g(f)-\omega([\lambda f, \lambda g]).$$

On the other hand, since $\Omega(\lambda f, \lambda g)=0$, we obtain

$$[\omega, \omega](\lambda f, \lambda g) = [f, g].$$

Hence

$$\omega([\lambda f, \lambda g]) = [f, g]+(\lambda f)g-(\lambda g)f$$

and, taking into account that $[\lambda f, \lambda g]$ is vertical, we deduce (1).

(2) follows by a similar device, considering that $[X^H, \lambda f]$ is vertical.

To prove (3), it suffices to recall that the horizontal component of $[X^H, Y^H]$ is $[X, Y]^H$ and, as a direct consequence of the structure equation,

$$\omega([X^H, Y^H]) = -2\Omega(X^H, Y^H).$$

Then,

$$[X^H, Y^H] = [X, Y]^H-2\lambda\Omega(X^H, Y^H).$$

We remark that if f and g are the constant functions A and B, respectively, $A, B\in\mathfrak{g}$, then the Proposition 3.1 gives the well-known formulas

(3.2) $$[\lambda A, \lambda B] = \lambda[A, B], \quad [X^H, \lambda B] = 0.$$

From now on, we return to the frame bundle of second order $\mathscr{F}^2M$ of a manifold M with a connection Γ of order 2. If X is a vector field on M with local expressions $X=X^i(\partial/\partial x^i)$ in a coordinate neighborhood U in M, then the local expression of X^H in $\mathscr{F}^2U$ with respect to the induced coordinates can be obtained by a direct computation and it is given by

(3.3) $$X^H = X^i\left\{\frac{\partial}{\partial x^i}-\Gamma^r_{il}X^l_j\frac{\partial}{\partial X^r_j}-(\Gamma^r_{il}X^l_{jk}+\Gamma^r_{ilm}X^l_jX^m_k)\frac{\partial}{\partial X^r_{jk}}\right\}$$

or

(3.4) $$X^H = X^iD_i,$$

with respect to the adapted frame.

From (2.9) and (3.3), one easily deduces that $\pi_1^2 X^H = X^H$, X^H being the horizontal lift of X to $\mathscr{F}M$ with respect to the induced connection $\bar{\Gamma}$.

Now, let F be a tensor field of type (1, 1) on M. We can define a function $F^\circ: \mathscr{F}M \to \mathrm{gl}(n)$ as follows: For any $p \in \mathscr{F}M$, $F^\circ(p)$ is the matrix representation of F_x with respect to p, $x = \pi(p)$. The function F induces a function on $\mathscr{F}^2M$, also denoted by F°, by putting $F^\circ = F^\circ \circ \pi_1^2$. If F has local components F_i^h in U, then we have in $\mathscr{F}^2U$

$$F^\circ = [F_t^h X_j^t Y_h^i],$$

and the corresponding fundamental vector field is given by

$$\lambda F^\circ = F_t^r X_j^t \frac{\partial}{\partial X_j^r} + F_t^h X_j^t Y_h^i X_{is}^r \frac{\partial}{\partial X_{js}^r} + F_t^h X_j^t Y_h^i X_{si}^r \frac{\partial}{\partial X_{sj}^r}. \tag{3.5}$$

Moreover, if $A \in \mathrm{gl}(n)$ and $\alpha \in S^2(n)$, we can consider the functions

$$F^\circ A: \mathscr{F}^2M \to \mathrm{gl}(n), \quad F^\circ\alpha: \mathscr{F}^2M \to S^2(n)$$

defined by

$$(F^\circ A)(p) = F^\circ(p)A, \quad (F^\circ\alpha)(p) = F^\circ(p) \circ \alpha, \quad p \in \mathscr{F}^2M.$$

The corresponding fundamental vector fields $\lambda(F^\circ A)$ and $\lambda(F^\circ\alpha)$ are locally expressed in $\mathscr{F}^2U$ as

$$\lambda(F^\circ A) = F_j^h X_s^j A_t^s \frac{\partial}{\partial X_t^h} + F_j^h X_s^j Y_h^r A_t^s X_{rk}^i \frac{\partial}{\partial X_{tk}^i} + F_j^h X_s^j Y_h^r A_t^s X_{kr}^i \frac{\partial}{\partial X_{kt}^i}, \tag{3.6}$$

$$\lambda(F^\circ\alpha) = F_j^h X_s^j \alpha_{mn}^s \frac{\partial}{\partial X_{mn}^h}; \tag{3.7}$$

where $A = A_j^i E_i^j$, $\alpha = \alpha_{jk}^i E_i^{jk}$, $\alpha_{jk}^i = \alpha_{kj}^i$.

The following formulas will be useful and can be obtained by a straightforward computation taking into account (3.3), (3.5), (3.6), (3.7) and Proposition 3.1:

$$\begin{aligned}
&[X^H, \lambda F^\circ] = \lambda(\nabla_X F)^\circ,\\
&[X^H, \lambda(F^\circ A)] = \lambda((\nabla_X F)^\circ A),\\
&[X^H, \lambda(F^\circ\alpha)] = \lambda((\nabla_X F)^\circ\alpha),\\
&[\lambda(F^\circ A), \lambda B] = \lambda(F^\circ[A, B]),\\
&[\lambda(F^\circ A), \lambda\alpha] = \lambda([F^\circ A, \alpha]) = \lambda\{F^\circ(A \circ \alpha) - \alpha \circ (I, F^\circ A) - \alpha \circ (F^\circ A, I)\},\\
&[\lambda(F^\circ A), \lambda(F^\circ B)] = \lambda((F^2)^\circ[A, B]),\\
&[\lambda(F^\circ A), \lambda(F^\circ\beta)] = \lambda\{(F^2)^\circ(A \circ \beta) - (F^\circ\beta) \circ (I, F^\circ A) - (F^\circ\beta) \circ (F^\circ A, I)\},\\
&[\lambda(F^\circ\alpha), \lambda B] = \lambda(F^\circ[\alpha, B]) = -\lambda\{F^\circ(B \circ \alpha) - F^\circ(\alpha \circ (I, B)) - F^\circ(\alpha \circ (B, I))\},\\
&[\lambda(F^\circ\alpha), \lambda\beta] = 0,\\
&[\lambda(F^\circ\alpha), \lambda(F^\circ\beta)] = 0,
\end{aligned} \tag{3.8}$$

for any vector field X, any tensor field F of type (1, 1) on M, any $A, B \in \mathrm{gl}(n)$, $\alpha, \beta \in S^2(n)$, where ∇ denotes the covariant derivative of the linear connection $\bar{\Gamma}$ induced from Γ.

Moreover, (1) in Proposition (3.1) can be equivalently written as follows:

$$[X^H, Y^H] = [X, Y]^H - 2\lambda\Omega_0(X^H, Y^H) - 2\lambda\Omega_1(X^H, Y^H).$$

Now, a simple computation shows that

$$2\lambda\Omega_0(X^H, Y^H) = \lambda(R(X, Y)^\circ),$$

R being the curvature tensor of $\bar{\Gamma}$. Then, we have

$$[X^H, Y^H] = [X, Y]^H - \lambda(R(X, Y)^\circ) - 2\lambda\Omega_1(X^H, Y^H), \tag{3.9}$$

for any vector fields X, Y on M.

4. Diagonal lifts of tensor fields

Let $u \in (R^n)^*$; then, there exist $u' \in \mathrm{gl}(n)^*$ and $u'' \in S^2(n)^*$ canonically associated to u and given by

$$u'(A) = \sum_{j=1}^{n} u(A_j), \quad A \in \mathrm{gl}(n), \quad \text{and} \quad u''(\alpha) = \sum_{j,k=1}^{n} u(\alpha_{jk}), \quad \alpha \in S^2(n),$$

where A_j (resp., α_{jk}) denotes the j^{th} column (resp., the (j, k)-column) of A (resp., α).

Let $u \in \mathrm{Hom}(R^n, R^n)$; then there exist

$$u' \in \mathrm{Hom}(\mathrm{gl}(n), \mathrm{gl}(n)) \quad \text{and} \quad u' \in \mathrm{Hom}(S^2(n), S^2(n))$$

canonically associated to u and given by

$$u'(A) = u \circ A, \quad A \in \mathrm{gl}(n), \quad \text{and} \quad u''(\alpha) = u \circ a, \quad \alpha \in S^2(n).$$

It is easy to show that if $\mathrm{rank}\, u = r$, then $\mathrm{rank}\, u' = rn$ and $\mathrm{rank}\, u'' = rn(n+1)/2$.

These two definitions of u' and u'' can be extended as follows: Let $u \in \otimes_s (R^n)^*$, $s \geqq 1$; then there exist $u' \in \otimes_s \mathrm{gl}(n)^*$ and $u'' \in \otimes_s S^2(n)^*$ canonically associated to u and given by

$$u'(A_1, \ldots, A_s) = \sum_{j=1}^{n} u((A_1)_j, \ldots, (A_s)_j), \quad A_1, \ldots, A_s \in \mathrm{gl}(n),$$

and

$$u''(\alpha_1, \ldots, \alpha_s) = \sum_{j,k=1}^{n} u((\alpha_1)_{jk}, \ldots, (\alpha_s)_{jk}), \quad \alpha_1, \ldots, \alpha_s \in S^2(n),$$

where $(A_i)_j$ (resp., $(\alpha_i)_{jk}$) denotes the j^{th} column (resp., the (j, k)-column) of A_i (resp., α_i).

In particular, if $s=2$, we have

$$u'(A, B) = \sum_{j=1}^{n} u(A_j, B_j), \quad A, B \in \mathrm{gl}(n),$$

$$u''(\alpha, \beta) = \sum_{j,k=1}^{n} u(\alpha_{jk}, \beta_{jk}), \quad \alpha, \beta \in S^2(n),$$

and it is easy to show that if u is symmetric (resp., skewsymmetric), then u' and u'' are also symmetric (resp., skewsymmetric). Moreover, if rank $u=r$, then rank $u'=rn$ and rank $u''=rn(n+1)/2$.

Now, let $u \in R^n \otimes (\otimes_s (R^n)^*)$, $s \geqq 1$; then there exist $u' \in \mathrm{gl}\,(n) \otimes (\otimes_s \mathrm{gl}\,(n)^*)$ and $u'' \in S^2(n) \otimes (\otimes_s S^2(n)^*)$ canonically associated to u such that the j^{th} column (resp., the (j, k)-column) of $u'(A_1, \dots, A_s)$ (resp., $u'(\alpha_1, \dots, \alpha_s)$) is $u((A_1)_j, \cdots, (A_s)_j)$ (resp., $u((\alpha_1)_{jk}, \dots, (\alpha_s)_{jk})$).

Let τ be an 1-form on M. For each $p \in \mathscr{F}^2 M$, we put

$$u_p = \tau_x \circ \pi_1^2(p), \tag{4.1}$$

where $\pi_1^2(p): R^n \to T_x M$, $x = \pi^2(p)$, is considered as a linear isomorphism. Then

Definition 4.1. The diagonal lift τ^D of τ to $\mathscr{F}^2 M$ is the 1-form given by

$$(\tau^D)_p(X) = u_p((\theta_{-1})_p X) + u'_p((\omega_0)_p X) + u''_p((\omega_1)_p X),$$

$X \in T_p(\mathscr{F}^2 M)$, $p \in \mathscr{F}^2 M$, where $u'_p \in \mathrm{gl}\,(n)^*$, $u''_p \in S^2(n)^*$ are the elements associated to $u_p \in (R^n)^*$ given by (4.1).

If $\tau = \tau_i \, dx^i$ is the local expression of τ in a coordinate neighborhood U, then the local expression of τ^D in $\mathscr{F}^2 U$ with respect to the adapted frame field is

$$\tau^D = \tau_i \eta^i + \sum_{j=1}^{n} \tau_i \eta^i_j + \sum_{j,k=1}^{n} \tau_i \eta^i_{jk}.$$

The definition above can be extended to an arbitrary covariant tensor field as follows. Let G be a tensor field on M of type $(0, s)$, $s \geqq 1$; for each $p \in \mathscr{F}^2 M$, we put

$$u_p = G_x \circ (\pi_1^2(p) \times \overset{s}{\dots} \times \pi_1^2(p)), \quad x = \pi_1^2(p). \tag{4.2}$$

Definition 4.2. The diagonal lift G^D of G to $\mathscr{F}^2 M$ is the tensor field of the same type given by

$$G^D(X_1, \dots, X_s) = u_p((\theta_{-1})_p X_1, \dots, (\theta_{-1})_p X_s) + u'_p((\omega_0)_p X_1, \dots, (\omega_0)_p X_s) + \\ + u''_p((\omega_1)_p X_1, \dots, (\omega_1)_p (X_s),$$

$X_1, \dots, X_s \in T_p(\mathscr{F}^2 M)$, $p \in \mathscr{F}^2 M$, where $u'_p \in \otimes_s \mathrm{gl}\,(n)^*$, $u''_p \in \otimes_s S^2(n)^*$ are the elements associated to $u_p \in \otimes_s (R^n)^*$, given by (4.2).

If $G_{j_1 \ldots j_s}$ are the local components of G in U, then

$$G^D = G_{j_1 \ldots j_s} \eta^{j_1} \otimes \ldots \otimes \eta^{j_s} + \sum_l \delta_l^{i_1} \ldots \delta_l^{i_s} G_{j_1 \ldots j_s} \eta_{i_1}^{j_1} \otimes \ldots \otimes \eta_{i_s}^{j_s} + \tag{4.3}$$

$$+ \sum_{l,m} \delta_l^{i_1} \delta_m^{j_1} \ldots \delta_l^{i_s} \delta_m^{j_s} G_{k_1 \ldots k_s} \eta_{i_1 j_1}^{k_1} \otimes \ldots \otimes \eta_{i_s j_s}^{k_s}$$

is the local expression of G^D with respect to the adapted frame field.

Now, let F be an arbitrary tensor field of type $(1, 1)$ on M. For each $p \in \mathscr{F}^2 M$, we put

$$u_p = (\pi_1^2(p))^{-1} \circ F_x \circ \pi_1^2(p), \quad x = \pi^2(p). \tag{4.4}$$

Then

Definition 4.3. The diagonal lift F^D of F to $\mathscr{F}^2 M$ is the tensor field of type $(1, 1)$ given by

$$(F^D)_p X = C(u_p((\theta_{-1})_p X)) + \lambda(u'_p((\omega_0)_p X)) + \lambda(u''_p((\omega_1)_p X)),$$

$X \in T_p(\mathscr{F}^2 M)$, $p \in \mathscr{F}^2 M$, where $u'_p \in \mathrm{Hom}(\mathrm{gl}(n), \mathrm{gl}(n))$, $u''_p \in \mathrm{Hom}(S^2(n), S^2(n))$ are the elements associated to $u_p \in \mathrm{Hom}(R^n, R^n)$ given by (4.4).

If F_j^h are the local components of F in U, then

$$F^D = F_j^h D_h \otimes \eta^j + \delta_j^i F_k^h D_h^i \otimes \eta_j^k + \delta_j^i \delta_l^k F_m^h D_{jl}^h \otimes \eta_{ik}^m \tag{4.5}$$

is the local expression of F^D with respect to the adapted frame field in $\mathscr{F}^2 U$.

Definition 4.3 can be extended as follows: Let F be a tensor field of type $(1, s)$, $s \geqq 1$; for each $p \in \mathscr{F}^2 U$, we put

$$u_p = (\pi_1^2(p))^{-1} \circ F_x \circ (\pi_1^2(p) \times \overset{s}{\ldots} \times \pi_1^2(p)), \quad x = \pi^2(p). \tag{4.6}$$

Then

Definition 4.4. The diagonal lift F^D of F to $\mathscr{F}^2 M$ is the tensor field of type $(1, s)$ given by

$$(F^D)_p(X_1, \ldots, X_s) = C(u_p((\theta_{-1})_p X_1, \ldots, (\theta_{-1})_p X_s)) +$$

$$+ \lambda(u'_p((\omega_0)_p X_1, \ldots, (\omega_0)_p X_s)) + \lambda(u''_p((\omega_1)_p X_1, \ldots, (\omega_1)_p X_s));$$

$X_1, \ldots, X_s \in T_p(\mathscr{F}^2 M)$, $p \in \mathscr{F}^2 M$, where $u'_p \in \mathrm{gl}(n) \otimes (\otimes_s \mathrm{gl}(n)^*)$, $u''_p \in S^2(n) \otimes (\otimes_s S^2(n)^*)$ are the elements associated to $u_p \in R^n \otimes (\otimes_s (R^n)^*)$ given by (4.6).

If $F_{j_1 \ldots j_s}^h$ are the local components of F in U, then

$$F^D = F_{j_1 \ldots j_s}^h D_h \otimes \eta^{j_1} \otimes \ldots \otimes \eta^{j_s} + \sum_{l,h} \delta_l^{i_1} \ldots \delta_l^{i_s} F_{j_1 \ldots j_s}^h D_h^l \otimes \eta_{i_1}^{j_1} \otimes \ldots \otimes \eta_{i_s}^{j_s} +$$

$$+ \sum_{l,m,h} \delta_l^{i_1} \delta_m^{j_1} \ldots \delta_l^{i_s} \delta_m^{j_s} F_{k_1 \ldots k_s}^h D_{lm}^h \otimes \eta_{i_1 j_1}^{k_1} \otimes \ldots \otimes \eta_{i_s j_s}^{k_s}$$

is the local expression of F^D in $\mathscr{F}^2 U$ with respect to the adapted frame field.

Now, let $\bar{\Gamma}$ be the linear connection on M induced from a connection Γ of order 2. The diagonal lifts to $\mathscr{F}M$ of tensor fields on M with respect to a linear connection on M have been considered in [1] by Cordero and de León. Actually, we can easily prove that the diagonal lifts of tensor fields to $\mathscr{F}^2M$ with Γ projects canonically to the diagonal lifts to $\mathscr{F}M$ with respect to $\bar{\Gamma}$.

5. Diagonal lifts of tensor fields of type (1, 1)

We shall now study the diagonal lift F^D of a tensor field F of type (1, 1) in more detail.

Proposition 5.1. *We have*

(1) $F^D X^H = (FX)^H$,

(2) $F^D(\lambda f) = \lambda(F^\circ f)$,

(3) $F^D(\lambda g) = \lambda(F^\circ g)$,

(4) $F^D(\lambda A) = \lambda(F^\circ A)$,

(5) $F^D(\lambda\alpha) = \lambda(F^\circ \alpha)$,

for any vector field X on M, any function f: $\mathscr{F}^2M \to \mathrm{gl}(n)$, any function g: $\mathscr{F}^2M \to S^2(n)$, any $A \in \mathrm{gl}(n)$ and any $\alpha \in S^2(n)$.

Proof. (1), (2) and (3) follow directly from (3.1), (3.4) and Definition 4.3 taking into account that $F^\circ(p) = u_p$; $p \in \mathscr{F}^2M$. (4) (resp., (5)) is a direct consequence of (2) (resp., (3)), when one considers the constant function A (resp., α).

From Proposition 5.1, we obtain

Proposition 5.2. *Let F, G be tensor fields of type* (1, 1) *on M and denote by I the identity tensor field. Then*

(1) $(FG)^D = F^D G^D$,

(2) $I^D = I$.

Proof. To prove (1) it suffices to check the identities

$$(FG)^D(X^H) = F^D(G^D(X^H));\quad (FG)^D(\lambda A) = F^D(G^D(\lambda A));\quad (FG)^D(\lambda\alpha) = F^D(G^D(\lambda\alpha));$$

for any vector field X on M, any $A \in \mathrm{gl}(n)$ and any $\alpha \in S^2(n)$. The first one follows from (1) in Proposition 5.1 and the other identities follow from (2), (3), (4) and (5) in Proposition 5.1 taking into account that $(FG)^\circ = F^\circ G^\circ$. On the other hand, we have

$$I^D X^H = (IX)^H = X^H, \quad I^D(\lambda A) = \lambda(I^\circ A) = \lambda A, \quad I^D(\lambda\alpha) = \lambda(I^\circ\alpha) = \lambda\alpha,$$

because I° is the constant function I. Thus, Proposition 5.2 is proved.

As a direct consequence of Proposition 5.2, we have

Proposition 5.3. *If $P(t)$ is a polynomial in one variable t, then*

$$(P(F))^D = P(F^D).$$

Corollary 5.4. *Let F be a tensor field of type* $(1, 1)$ *on M. Then, if F defines on M a polynomial structure of* rank r *and structural polynomial* $P(t)=0$, *its diagonal lift F^D defines on $\mathscr{F}^2M$ a polynomial structure of* rank $r(1+n+n(n+1)/2)$ *and with the same structural polynomial. In particular, if F is an almost complex structure (resp., an f-structure of* rank r*) on M, then F^D is an almost complex structure (resp., an f-structure of* rank $r(1+n+n(n+1)/2)$*) on $\mathscr{F}^2M$.*

Denote by N_{F^D} and N_F the Nijenhuis tensor of F^D and F, respectively. Thus, taking into account the definition of the Nijenhuis tensor, the formulas (3.8) and (3.9) and Proposition 5.1, we find by a straightforward computation the following identities:

$$\begin{aligned} N_{F^D}(X^H, Y^H) = {} & (N_F(X, Y))^H - \lambda((R(FX, FY) - FR(FX, Y) - FR(X, FY) + \\ & + F^2R(X, Y))^\circ) - 2\lambda(\Omega_1((FX)^H, (FY)^H) - F^\circ\Omega_1((FX)^H, Y^H) - \\ & - F^\circ\Omega_1(X^H, (FY)^H) + (F^2)^\circ\Omega_1(X^H, Y^H)),\end{aligned}$$

$$N_{F^D}(X^H, \lambda B) = \lambda((\nabla_{FX}F - F\nabla_X F)^\circ B), \quad N_{F^D}(X^H, \lambda\beta) = \lambda((\nabla_{FX}F - F\nabla_X F)^\circ \beta),$$

$$N_{F^D}(\lambda A, \lambda B) = N_{F^D}(\lambda A, \lambda\beta) = N_{F^D}(\lambda\alpha, \lambda\beta) = 0,$$

for any vector fields X, Y on M, any $A, B \in \mathrm{gl}(n)$ and any $\alpha, \beta \in S^2(n)$. Therefore, we have

Proposition 5.5. *Let F be a tensor field on M of type* $(1, 1)$ *and F^D its diagonal lift to $\mathscr{F}^2M$. Then the condition $N_{F^D}=0$ is equivalent to the conditions*

$$N_F = 0, \quad F\nabla_X F - \nabla_{FX} F = 0,$$

$$R(FX, FY) - FR(FX, Y) - FR(X, FY) + F^2R(X, Y) = 0,$$

$$\Omega_1((FX)^H, FY)^H) - F^\circ\Omega_1((FX)^H, Y^H) - F^\circ\Omega_1(X^H, (FY)^H) + (F^2)^\circ\Omega_1(X^H, Y^H) = 0,$$

for arbitrary vector fields X, Y on M. The three last conditions can be equivalently *written as*

$$F_i^k \nabla_k F_j^h - F_k^h \nabla_i F_j^h = 0,$$

$$R_{klm}^h F_j^l F_i^m - R_{kli}^m F_j^l F_m^h - R_{kjl}^m F_i^l F_m^h + R_{kji}^l F_l^m F_m^h = 0,$$

$$R_{kjlm}^h F_s^l F_t^m - R_{kjlt}^i F_s^l F_i^h - R_{kjsm}^i F_i^h F_t^m + R_{kjst}^i F_r^h F_i^r = 0,$$

where F_j^h are the local components of F.

To obtain some meaningful formulas on Lie derivatives, let us recall that the Lie derivative $\mathscr{L}_{\tilde{X}}\tilde{F}$ of a tensor field $\tilde{F}$ of type $(1, 1)$ with respect to a vector field $\tilde{X}$ is defined by

$$(\mathscr{L}_{\tilde{X}}\tilde{F})(\tilde{Y}) = [\tilde{X}, \tilde{F}\tilde{Y}] - \tilde{F}[\tilde{X}, \tilde{Y}],$$

$\tilde{Y}$ being an arbitrary vector field. Thus, taking into account (3.8), (3.9) and Proposition 5.1, we obtain

$$(\mathscr{L}_{X^H} F^D)(Y^H) = ((\mathscr{L}_X F)(Y))^H - \lambda((R(X, FY) - FR(X, Y))^\circ) -$$
$$-2\lambda(\Omega_1(X^H, (FY)^H) - F^\circ \Omega_1(X^H, Y^H)),$$
$$(\mathscr{L}_{X^H} F^D)(\lambda A) = \lambda((\nabla_X F)^\circ A), \quad (\mathscr{L}_{X^H} F^D)(\lambda\alpha) = ((\nabla_X F)^\circ \alpha),$$

for any vector fields X, Y on M, any $A \in \mathrm{gl}(n)$ and any $\alpha \in S^2(n)$. Thus, we have

Proposition 5.6. *Let X be a vector field and F a tensor field of type* $(1, 1)$ *on M. Then the condition $\mathscr{L}_{X^H} F^D = 0$ is equivalent to the conditions*

$$\mathscr{L}_X F = 0, \quad \nabla_X F = 0, \quad R(X, FY) - FR(X, Y) = 0,$$
$$\Omega_1(X^H, (FY)^H) - F^\circ \Omega_1(X^H, Y^H) = 0,$$

for any vector field Y on M. The two last conditions can be equivalently written as

$$X^j(R_{kjl}^h F_i^l - R_{kji}^l F_l^h) = 0, \quad X^m(R_{kjlm}^h F_i^l - R_{kjim}^l F_l^h) = 0,$$

where X^j and F_j^h are the local components of X and F, respectively.

If we next take into account (3.7) and Proposition 5.1, we find

$$(\mathscr{L}_{\lambda A} F^D)(Y^H) = (\mathscr{L}_{\lambda A} F^D)(\lambda B) = (\mathscr{L}_{\lambda A} F^D)(\lambda\beta) = 0,$$
$$(\mathscr{L}_{\lambda\alpha} F^D)(Y^H) = (\mathscr{L}_{\lambda\alpha} F^D)(\lambda B) = (\mathscr{L}_{\lambda\alpha} F^D)(\lambda\beta) = 0,$$

for any vector field Y on M, any $A, B \in \mathrm{gl}(n)$ and any $\alpha, \beta \in S^2(n)$. Thus, we have

Proposition 5.7. *Let F be a tensor field on M of type* $(1, 1)$. *Then* $\mathscr{L}_{\lambda A} F^D = \mathscr{L}_{\lambda\alpha} F^D = 0$, *for any* $A \in \mathrm{gl}(n)$ *and any* $\alpha \in S^2(n)$.

6. Diagonal lifts of tensor fields of type (0, 2)

Let G be a tensor field of type $(0, 2)$ on M. Particularizing in this case Definition 4.2 and (4.3), we have that the diagonal lift G^D of G to $\mathscr{F}^2M$ is a tensor field of type $(0, 2)$ on $\mathscr{F}^2M$ with local expression in $\mathscr{F}^2U$

$$G^D = G_{ij}\eta^i\otimes\eta^j+\delta^{kl}G_{ij}\eta^i_k\otimes\eta^j_l+\delta^{km}\delta^{ln}G_{ij}\eta^i_{kl}\otimes\eta^j_{mn}, \tag{6.1}$$

G_{ij} being the local components of G in U.

By using (6.1), one easily deduces that if G has constant rank r, then G^D has constant rank $r(1+n+n(n+1)/2)$. Thus, we have

Proposition 6.1. (1) *If G is a Riemannian metric on M, then G^D is a Riemannian metric on $\mathscr{F}^2M$.*

(2) *If G is an almost symplectic form on M, then G^D is an almost symplectic form on $\mathscr{F}^2M$.*

Let us now introduce two new definitions: Let $A, B\in\mathrm{gl}(n)$, $\alpha, \beta\in S^2(n)$ and let G be a tensor field of type $(0, 2)$ on M with local components G_{ij}; then

$$G^\circ(A, B) = \delta^{rs}A^k_rB^l_sX^i_kX^j_lG_{ij} \tag{6.2}$$

and

$$G^\circ(\alpha, \beta) = \delta^{rt}\delta^{sn}\alpha^k_{rs}\beta^l_{tn}X^i_kX^j_lG_{ij} \tag{6.3}$$

are globally well-defined functions on $\mathscr{F}^2M$, where

$$A = A^i_jE^j_i,\quad B = B^i_jE^j_i,\quad \alpha = \alpha^i_{jk}E^{jk}_i,\quad \beta = \beta^i_{jk}E^{jk}_i.$$

The following formulas are easily obtained:

$$\begin{gathered} G^D(\lambda A, \lambda B) = G^\circ(A, B),\quad G^D(\lambda A, \lambda\beta) = G^D(\lambda\beta, \lambda A) = 0,\\ G^D(\lambda A, X^H) = G^D(X^H, \lambda A) = G^D(\lambda\alpha, X^H) = G^D(X^H, \lambda\alpha) = 0,\\ G^D(\lambda\alpha, \lambda\beta) = G^\circ(\alpha, \beta),\quad G^D(X^H, Y^H) = (G(X, Y))^V, \end{gathered} \tag{6.4}$$

for any $A, B\in\mathrm{gl}(n)$, $\alpha, \beta\in S^2(n)$ and arbitrary vector fields X, Y on M, where $f^V=f\circ\pi^2$, for any function f on M.

Next, we shall compute the Lie derivatives of G^D with respect to vector fields λA, $\lambda\alpha$ or X^H on $\mathscr{F}^2M$. To do this, let us recall that the Lie derivative $\mathscr{L}_{\tilde X}\tilde G$ of a tensor field $\tilde G$ of type $(0, 2)$ with respect to a vector field $\tilde X$ is defined by

$$(\mathscr{L}_{\tilde X}\tilde G)(\tilde Y, \tilde Z) = \tilde X(\tilde G(\tilde Y, \tilde Z))-\tilde G([\tilde X, \tilde Y], \tilde Z)-\tilde G(\tilde Y, [\tilde X, \tilde Z]),$$

$\tilde Y$ and $\tilde Z$ being arbitrary vector fields.

Proposition 6.2. *For any* $A, B, C \in \mathrm{gl}(n)$, $\alpha, \beta \in S^2(n)$ *and* X, Y *vector fields on* M, *we have*

(1) $(\mathscr{L}_{\lambda A} G^D)(X^H, \lambda B) = (\mathscr{L}_{\lambda A} G^D)(\lambda B, X^H) = 0$,

(2) $(\mathscr{L}_{\lambda A} G^D)(X^H, \lambda\beta) = (\mathscr{L}_{\lambda A} G^D)(\lambda\beta, X^H) = 0$,

(3) $(\mathscr{L}_{\lambda A} G^D)(X^H, Y^H) = 0$,

(4) $(\mathscr{L}_{\lambda A} G^D)(\lambda\beta, \lambda C) = G^\circ(B(A+A^t), C)$,

(5) $(\mathscr{L}_{\lambda A} G^D)(\lambda B, \lambda\beta) = (\mathscr{L}_{\lambda A} G^D)(\lambda\beta, \lambda B) = 0$,

(6) $(\mathscr{L}_{\lambda A} G^D)(\lambda\alpha, \lambda\beta) = G^\circ(\alpha \circ (I, A+A^t), \beta) + G^\circ(\alpha, \beta^\circ(I, A+A^t))$,

where A^t *denotes the transpose of* A.

Proof. (1), (2), (3) and (5) follow directly from (3.2), (3.9) and (6.4). (4) and (6) follow by a direct computation using (6.2) and (6.3).

Corollary 6.3. *Let* G *be a Riemannian metric (resp., an almost symplectic form) on* M. *Then the fundamental vector field* λA *on* $\mathscr{F}^2 M$ *is a Killing vector field (resp., an infinitesimal automorphism) of* $(\mathscr{F}^2 M, G^D)$ *if and only if* $A + A^t = 0$, *that is, if and only if* A *is skewsymmetric.*

Proposition 6.4. *For any* $A, B \in \mathrm{gl}(n)$, $\alpha, \beta, \gamma \in S^2(n)$ *and* X, Y *vector fields on* M, *we have*

(1) $(\mathscr{L}_{\lambda\alpha} G^D)(X^H, \lambda B) = (\mathscr{L}_{\lambda\alpha} G^D)(\lambda B, X^H) = 0$,

(2) $(\mathscr{L}_{\lambda\alpha} G^D)(X^H, \lambda\beta) = (\mathscr{L}_{\lambda\alpha} G^D)(\lambda\beta, X^H) = 0$,

(3) $(\mathscr{L}_{\lambda\alpha} G^D)(X^H, Y^H) = 0$,

(4) $(\mathscr{L}_{\lambda\alpha} G^D)(\lambda A, \lambda B) = 0$,

(5) $(\mathscr{L}_{\lambda\alpha} G^D)(\lambda A, \lambda\beta) = (\mathscr{L}_{\lambda\alpha} G^D)(\lambda\beta, \lambda A) = 0$,

(6) $(\mathscr{L}_{\lambda\alpha} G^D)(\lambda\beta, \lambda\gamma) = 0$.

Proof. It follows by a direct computation in a way similar to that of the proof of Proposition 6.2.

Corollary 6.5. *Let* G *be a Riemannian metric (resp., an almost symplectic form) on* M. *Then the fundamental vector field* $\lambda\alpha$ *on* $\mathscr{F}^2 M$ *is always a Killing vector field (resp., an infinitesimal automorphism) of* $(\mathscr{F}^2 M, G^D)$.

From Corollary 6.3 and Corollary 6.5, we have

Corollary 6.6. *Let* G *be a Riemannian metric (resp., an almost symplectic form) on* M. *Then the fundamental vector field* $\lambda(A, \alpha)$ *on* $\mathscr{F}^2 M$ *is a Killing vector* *field (resp., an infinitesimal automorphism) of* $(\mathscr{F}^2M, G^D)$ *if and only if* $A+A^t=0$, *that is, A is skewsymmetric.*

Proposition 6.7. *For any* $A, B\in \mathrm{gl}(n)$, $\alpha, \beta\in S^2(n)$ *and vector fields X, Y, Z on M, we have*

(1) $(\mathscr{L}_{X^H}G^D)(Y^H, Z^H) = ((\mathscr{L}_X G)(Y, Z))^V$,

(2) $(\mathscr{L}_{X^H}G^D)(Y^H, \lambda A) = G^D(\lambda R(X, Y)^\circ, \lambda A)$,

$(\mathscr{L}_{X^H}G^D)(\lambda A, Y^H) = G^D(\lambda A, \lambda R(X, Y)^\circ)$,

(3) $(\mathscr{L}_{X^H}G^D)(Y^H, \lambda\alpha) = 2G^D(\lambda\Omega_1(X^H, Y^H), \lambda\alpha)$,

$(\mathscr{L}_{X^H}G^D)(\lambda\alpha, Y^H) = 2G^D(\lambda\alpha, \lambda\Omega_1(X^H, Y^H))$,

(4) $(\mathscr{L}_{X^H}G^D)(\lambda A, \lambda B) = (\nabla_X G)^\circ(A, B)$,

(5) $(\mathscr{L}_{X^H}G^D)(\lambda A, \lambda\alpha) = (\mathscr{L}_{X^H}G^D)(\lambda\alpha, \lambda A) = 0$,

(6) $(\mathscr{L}_{X^H}G^D)(\lambda\alpha, \lambda\beta) = (\nabla_X G)^\circ(\alpha, \beta)$.

Proof. The proof follows by a straightforward computation from (3.2), (3.9), (6.2), (6.3) and (6.4).

Corollary 6.8. *Let X be a vector field and G a tensor field of type* (0, 2) *on M. Then the condition* $\mathscr{L}_{X^H}G^D=0$ *is equivalent to the conditions*

$$\mathscr{L}_X G = 0, \quad R(X, \cdot) = 0, \quad \nabla_X G = 0, \quad i_{X^H}\Omega_1 = 0,$$

where $R(X, \ )$ *denotes the tensor field of type* (1, 2) *on M given by* $R(X, \ )(Y, Z)=$ $=R(X, Y)Z$, *for any vector fields Y, Z on M.*

Let us now suppose that the induced connection ∇ on M from a connection Γ of order 2 is the Riemannian connection of a Riemannian metric G on M. Then, we have

Theorem 6.9. *If the horizontal lift* X^H *to* $\mathscr{F}^2M$ *of a vector field X on M is a Killing vector field in* $(\mathscr{F}^2M, G^D)$, *then X is a Killing vector field in* (M, G). *Conversely, suppose that* Γ *is partially flat and X is a Killing vector field with vanishing second covariant derivative in* (M, G); *then* X^H *is a Killing vector field in* $(\mathscr{F}^2M, G^D)$.

Proof. We only need to prove the converse. Indeed, if X is a Killing vector field in (M, G), then $\mathscr{L}_X G=0$ and, hence $\mathscr{L}_X\nabla=0$. Therefore, $R(X, Y)=-$ $-\nabla_Y(\nabla X)$. Moreover, since $(\nabla_Y(\nabla X))(Z)=(\nabla^2 X)(Z, Y)$, we have $R(X, Y)Z=$ $=-(\nabla^2 X)(Z, Y)$.

To end this section, we shall consider an almost symplectic form G on M. The following set of formulas is easily obtained by a straightforward computation:

$$
\begin{aligned}
&dG^D(X^H, Y^H, Z^H) = \{dG(X, Y, Z)\}^V,\\
&dG^D(X^H, Y^H, \lambda C) = (1/3)G^D(\lambda(R(X, Y)^\circ), \lambda C),\\
&dG^D(X^H, Y^H, \lambda\gamma) = (2/3)G^D(\lambda(\Omega_1(X^H, Y^H)), \lambda\gamma),\\
&dG^D(X^H, \lambda B, \lambda C) = (1/3)\{(\nabla_X G)^\circ(B, C)\},\\
&dG^D(X^H, \lambda B, \lambda\gamma) = 0,\\
(6.5)\qquad &dG^D(X^H, \lambda\beta, \lambda\gamma) = (1/3)\{(\nabla_X G)^\circ(\beta, \gamma)\},\\
&dG^D(\lambda A, \lambda B, \lambda C) = 0,\\
&dG^D(\lambda A, \lambda B, \lambda\gamma) = 0,\\
&dG^D(\lambda A, \lambda\beta, \lambda\gamma) = (1/3)\{G^\circ(\beta\circ(I, A)+\beta\circ(A, I), \gamma)+G^\circ(\beta, \gamma\circ(I, A)+\gamma\circ(A, I))\},\\
&dG^D(\lambda\alpha, \lambda\beta, \lambda\gamma) = 0
\end{aligned}
$$

for any vector fields X, Y, Z on M, any $A, B \in \mathrm{gl}(n)$ and any $\alpha, \beta, \gamma \in S^2(n)$.

Then, we have

Proposition 6.10. *The almost symplectic form G is never closed; consequently, the almost symplectic manifold* $(\mathscr{F}^2M, G^D)$ *is never sympletic.*

Proof. In fact, if we take $A=I$ in (6.5), we obtain

$$dG^D(\lambda I, \lambda\beta, \lambda\gamma) = (4/3)G^\circ(\beta, \gamma) \quad \text{for any} \quad \beta, \gamma \in S^2(n).$$

7. The frame bundle of order 2 of an almost Hermitian manifold

Let M be an m-dimensional manifold, J a tensor field on M of type (1, 1) such that $J^2=-I$ and G a Riemannian metric on M such that $G(JX, JY)=G(X, Y)$ for any vector fields X, Y on M; then, (M, J, G) is said to be an almost Hermitian manifold.

Let Γ be a connection of order 2 on M. Since $(J^D)^2=-I$ from Proposition 5.3 and G^D is a Riemannian metric on $\mathscr{F}^2M$ from Proposition 6.1, we have

Proposition 7.1. $(\mathscr{F}^2M, J^D, G^D)$ *is an almost Hermitian manifold.*

Proof. It suffices to check the identity

$$G^D(J^D\tilde{X}, J^D\tilde{Y}) = G^D(\tilde{X}, \tilde{Y})$$

in the following three particular cases:

(1) $\tilde{X}=X^H$, $\tilde{Y}=Y^H$, X, Y being arbitrary vector fields on M. The identity follows taking into account Proposition 5.1 and (6.4).

(2) $\tilde{X}=X^H$ and $\tilde{Y}=\lambda(A, \alpha)$ for any vector field X on M and any $A\in \mathrm{gl}\,(n)$, $\alpha\in S^2(n)$. In this case, both members of the identity vanish from Proposition 5.1 and (6.4).

(3) $\tilde{X}=\lambda(A, \alpha)$, $\tilde{Y}=\lambda(B, \beta)$ for any $A, B\in \mathrm{gl}\,(n)$, $\alpha, \beta\in S^2(n)$. The result follows by a straightforward computation taking into account Proposition 5.1, (6.4) and the Hermitian character of G.

Let Φ be the Kaehler form of (M, J, G), that is, Φ is the 2-form on M given by

$$\Phi(X,Y) = G(X, JY),$$

for any vector fields X, Y on M. Then, we have

Proposition 7.2. *The Kaehler form of the almost Hermitian manifold $(\mathcal{F}^2M, J^D, G^D)$ is the diagonal lift Φ^D of the Kaehler form Φ of (M, J, G).*

Proof. The proof is similar to that of Proposition 7.1 and is left to the reader.

Now, let us recall that an almost Hermitian manifold (M, J, G) is said to be 1) Hermitian, if $N_J=0$; 2) almost Kaehler, if $d\Phi=0$; 3) Kaehler, if $N_J=0$ and $d\Phi=0$. Moreover, it is well known that the Kaehler form Φ of (M, J, G) is almost symplectic. Thus, we deduce

Theorem 7.3. *Let (M, J, G) be an almost Hermitian manifold. Then*

(1) *$(\mathcal{F}^2M, J^D, G^D)$ is never an almost Kaehler manifold.*

(2) *Moreover, if (M, J, G) is a Kaehler manifold and its Riemannian connection is the induced linear connection on M from Γ, then $(\mathcal{F}^2M, J^D, G^D)$ is an Hermitian manifold if Γ has zero curvature.*

Proof. (1) is a direct consequence of Proposition 6.10, and (2) follows easily from Proposition 5.5.

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